MATHEMATICS

ON A CONNECTION BETWEEN TWO TECHNIQUES FOR THE NUMERICAL TRANSFORMATION OF SLOWLY CONVERGENT SERIES [1])

BY

P. WYNN

(Communicated by Prof. A. VAN WIJNGAARDEN at the meeting of September 30, 1961)

1. BICKLEY and MILLER have proposed difference equation methods for obtaining converging factors for two types of slowly convergent series. The two types of series

$$S \sim \sum_{n=0}^{\infty} u_n \tag{1}$$

are those for which

$$\lim_{n\to\infty} \left\{\frac{u_{n+1}}{u_n}\right\} = x \neq 1 \tag{2}$$

and a series expansion of the form

$$\frac{u_{n+1}}{u_n} = x\left\{1 + \frac{A_1}{n} + \frac{A_2}{n^2} + \dots\right\} \tag{3}$$

may be given, and those for which a development of the form

$$\frac{u_{n+1}}{u_n} = 1 - \frac{A_1}{n} + \frac{A_2}{n^2} + \dots \tag{4}$$

exists.

2. The converging factor, defined by

$$u_n C_n \sim \sum_{s=0}^{\infty} u_{n+s} \tag{5}$$

is, for the series of the first type, expanded as

$$C_n = \alpha_0 + \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \dots. \tag{6}$$

The coefficients in (6) are determined [1] by substitution in the difference equation

$$u_n C_n = u_n + u_{n+1} C_{n+1} \tag{7}$$

or

$$\begin{cases} \alpha_0 - 1 + \dfrac{\alpha_1}{n} + \dfrac{\alpha_2}{n^2} + \dots + \dfrac{\alpha_s}{n^s} + \dots = \\ x\left\{1 + \dfrac{A_1}{n} + \dfrac{A_2}{n^2} + \dots + \dfrac{A_s}{n^s} + \dots\right\}\left\{\alpha_0 + \dfrac{\alpha_1}{n} + \dfrac{\alpha_2 - \alpha_1}{n^2} + \dots + \dfrac{\triangle^s \alpha_1}{n^{s+1}} + \dots\right\}. \end{cases} \tag{8}$$

---

1) Communication MR 38 of the Computation Department of the Mathematical Centre at Amsterdam.

There follow

$$\text{(9)} \qquad \alpha_0 = \frac{1}{1-x}, \ \alpha_1 = \frac{xA_1}{(1-x)^2}, \ \alpha_2 = \{x(1-x)A_2 + x^2A_1{}^2 - xA_1\}\frac{1}{(1-x)^3}, \ \ldots.$$

This method can be applied to the transformation of the asymptotic series

$$\text{(10)} \qquad ze^zEi(-z) \sim \sum_{n=0}^{\infty} n!(-z)^{-n}.$$

Letting

$$\text{(11)} \qquad z = (n+h)\beta \qquad \beta = e^{-i\theta}$$

and taking h to be small (so that the transformation is applied at the smallest term) there follows

$$\text{(12)} \qquad \frac{u_{n+1}}{u_n} = -\left\{\frac{n+1}{n+h}\right\} e^{-i\theta} = -e^{-i\theta}\{1 + \sum_{s=0}^{\infty}(-h)^s(1-h)n^{-s-1}\}$$

from which coefficients in the appropriate converging factor may be derived.

The accuracy obtainable by use of the series (10) may also be improved if $|z|$ is reasonably large by applying the Euler transformation

$$\text{(13)} \qquad \sum_{s=0}^{\infty}(-\beta)^s u_{n+s} \sim \frac{1}{1+\beta}\sum_{s=0}^{\infty}\left(\frac{-\beta}{1+\beta}\right)^s \triangle^s u_n$$

to the series starting with the smallest term.

Barkeley Rosser [2] has pointed out that if the terms obtained by applying (13) to (10) in the manner explained are expanded in inverse powers of n (leaving a factor $(-1)^n n! z^{-n}$ outside the summation), and rearranged, there results a series of the form (6) identical to that obtained by the Bickley–Miller method; (in the event, equivalent to a series given by Airey which may most expeditiously be derived by the Bickley–Miller method). The first purpose of this note is to point out that this equivalence is general and not confined to a specific example.

The generalised Euler transformation may be written

$$\text{(14)} \qquad \sum_{s=0}^{\infty} x^s u_{n+s} \sim \frac{1}{1-x}\left\{u_n + \frac{x}{1-x}\underset{s}{\triangle} u_n + \left(\frac{x}{1-x}\right)^2 \underset{s}{\triangle^2} u_n + \ldots\right\}.$$

But if (3) obtains, then

$$\text{(15)} \qquad \underset{s}{\triangle^r} u_n = u_n \underset{s}{\triangle^r}\{\phi_0\}$$

where

$$\text{(16)} \qquad \phi_s = \prod_{i=0}^{s}\left\{1 + \frac{A_1}{(n+i+1)} + \frac{A_2}{(n+i+1)^2} + \ldots\right\}.$$

Evidently

$$(17)\qquad \begin{cases} \underset{s}{\triangle}^0 u_n = u_n \\ \underset{s}{\triangle}^1 u_n = u_n \left\{ \dfrac{A_1}{n} + \dfrac{A_2 - A_1}{n^2} + \dots \right\} \\ \underset{s}{\triangle}^2 u_n = u_n \left\{ -\dfrac{A_1(1+A_2)}{n^2} + 0(n^{-3}) \right\} \end{cases}$$

and

$$(18)\qquad \underset{s}{\triangle}^r u_n = u_n\{0(n^{-r})\}$$

Substituting the results (17) in (14) and rearranging, there follows

$$(19)\qquad \sum_{s=0}^{\infty} u_{n+s} x^s \sim u_n F_n$$

where

$$(20)\qquad F_n \sim \frac{1}{(1-x)} + \frac{x}{1-x} \cdot \frac{A_1}{n} + \{A_2 x(1-x) + x^2 A_1^2 - A_1 x\} \frac{1}{n^2} + \dots$$

in agreement with (9).

It still remains to be shown that F_n and C_n are in fact the same function. However they both satisfy the same first order linear difference equation of the form (7) and are therefore linearly dependent; inspection of the leading coefficients in (9) and (20) shows that they are equal.

3. The converging factor expansion appropriate to series for which relation (4) obtains, is

$$(21)\qquad C_n = \alpha_{-1} n + \alpha_0 + \alpha_1 n^{-1} + \dots.$$

Substitution in (7) then yields [4]

$$(22)\qquad \begin{cases} \alpha_{-1}\, n + \alpha_0 - 1 + \dfrac{\alpha_1}{n} + \dots + \dfrac{\alpha_s}{n^s} + \dots \\ = \left\{ \alpha_{-1} n + \alpha_{-1} + \alpha_0 + \dfrac{\alpha_1}{n} + \dots + \dfrac{\triangle^s \alpha_1}{n^{s+1}} + \dots \right\} \left\{ 1 - \dfrac{A_1}{n} + \dfrac{A_2}{n^2} + \dfrac{A_3}{n^3} + \dots \right\} \end{cases}$$

from which the coefficients

$$(23)\qquad \alpha_{-1} = \frac{1}{1-A_1},\quad \alpha_0 = \frac{A_2 + A_1 - A_1^2}{A_1(1-A_1)},\quad \alpha_1 = \frac{A_1 A_3 + A_2^2}{(A_1+1)A_1(1-A_1)},\ \dots$$

may recursively be obtained. (The notation and working adopted in this and the previous example are slightly at variance with that occurring in Bickley and Miller's original treatment).

Now there is a further transformation suitable for accelerating the convergence of slowly convergent series.

It is

$$
(24)\quad \begin{cases} v_n + \dfrac{x}{x-y+1} v_{n+1} + \dfrac{x(x+1)}{(x-y+1)(x-y+2)} v_{n+2} + \dots \\[2ex] \qquad = (y-x)\left\{\dfrac{v_n}{y} - \dfrac{x}{y(y+1)} \triangle v_n + \dfrac{x(x+1)}{y(y+1)(y+2)} \triangle^2 v_n - \dots\right\} \end{cases}
$$

and is applied to the transformation of the series

$$
(25)\qquad R_n \sim u_n + u_{n+1} + u_{n+2} + \dots
$$

by writing successively

$$
(26)\quad \begin{cases} v_n = u_n \\[1ex] v_{n+1} = \dfrac{x-y+1}{x} u_{n+1} \\[2ex] v_{n+2} = \dfrac{(x-y+1)(x-y+2)}{x(x+1)} u_{n+2} \\[1ex] \dots\dots\dots\dots\dots \end{cases}
$$

for appropriately chosen values of x and y. If the sequence v_{n+s} $s=0, 1, \dots$ so derived remains approximately constant, the sequence $\triangle^r v_n$ $r=0, 1, \dots$ diminishes rapidly, and the numerical convergence of the series upon the right hand side of equation (24) is more rapid than that of the series upon the left. This transformation, together with the Euler transformation is one of a family of transformations of Euler–Gudermann type, of which a comparative survey is given in [5].

By writing

$$
(27)\qquad x = n - A_1 + 1, \qquad y = 1 - A_1
$$

(24 becomes

$$
(28)\qquad v_n + \frac{n-A_1+1}{n+1} v_{n+1} + \frac{(n-A_1+1)(n-A_1+2)}{(n+1)(n+2)} v_{n+2} + \dots
$$

$$
(29)\qquad = \frac{n}{A_1-1} v_n + \frac{n(n-A_1+1)}{(A_1-1)(A_2-2)} \triangle v_n + \frac{n(n-A_1+1)(n-A_1+2)}{(A_1-1)(A_1-2)(A_1-3)} \triangle^2 v_n + \dots
$$

and relations (26) become

$$
(30)\qquad v_{n+r} = \prod_{s=0}^{r} \phi_s u_n \qquad r=0, 1, \dots
$$

where

$$
(31)\quad \begin{cases} \phi_0 = 1 \\[1ex] \phi_s = \dfrac{n+s}{n-A_1+s}\left\{1 - \dfrac{A_1}{n+s} + \dfrac{A_2}{(n+s)^2} + \dots\right\} \qquad s=1, 2, \dots. \end{cases}
$$

But, as is easily verified

$$
(32)\qquad \phi_s = 1 + \frac{A_1 - A_1{}^2 + A_2}{n^2} + \frac{A_3 - 2sA_2 + A_1A_2}{n^3} + \dots + \frac{p_r(s)}{n^{r+2}}
$$

where $\phi_r(s)$ is a polynomial of degree r in s. Thus

(33) $$\triangle^r v_n = 0(n^{-r-1}) \qquad r = 1, 2, \dots$$

and the only linear term in n in F_n is contributed by the first term in expression (28), or

(34) $$F_n = \frac{n}{1-A_1} + 0(1).$$

Again the F_n and C_n of this case satisfy the same first order linear difference equation, have the same leading term, and are therefore the same functions.

4. The final point to be made and indeed it is the purpose for which this note was written is this: numerical experience shows that the Bickley–Miller difference equation techniques are extremely powerful when applied to a suitable example, but in most practically meaningful cases the coefficients A_s $s = 1, 2, \dots$ in (3) or (4) are difficult to determine. The results of this note show how numerically equivalent techniques may be applied, which demand only the previous determination of

(35) $$x = \lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right)$$

or

(36) $$A_1 = \lim_{n\to\infty} n\left(1 - \frac{u_{n+1}}{u_n}\right)$$

Remark. It is interesting to note that the order relationships (18) and (33) show why in certain cases the Euler transformation is successful and the transformation (28) is not. The result (18) implies that successive terms in the Euler transformation behave like a power series with argument $-x/\{n(1+x)\}$; the result (33) implies however that the first term in C_n is $0(n)$ (actually $n/(A_1-1)$), but that the remaining terms are of the same order of magnitude and do not rapidly decrease. Series for which relation (4) obtains are more favourably treated by means of the ϱ-algorithm [6].

REFERENCES

1. Bickley, W. G. and J. C. P. Miller, The Numerical Transformation of Slowly Convergent Series, unpublished memoir.
2. Barkley Rosser, J., Transformations to Speed the Convergence of Series, Journ. Res. Nat. Bur. Stand. **46**, 56 (1951).
3. Airey, J. R., The Converging Factor in Asymptotic Series and the Calculation of Bessel, Laguerre and other Functions, Phil. Mag. (7), **24**, 521 (1937).
4. Bickley, W. G. and J. C. P. Miller, The Numerical Summation of Slowly Convergent Series of Positive Terms, Phil. Mag. Ser. 7, vol. XXII, 754 (1936).

5. WYNN, P., The Numerical Transformation of Slowly Convergent Series by Methods of Comparison, Chiffres, to appear.

6. ———, On a Procrustean Technique for the Numerical Transformation of Slowly Convergent Sequences and Series, Proc. Camb. Phil. Soc., **52**, Part 4, 663 (1956).